

# On non-local representations of the ageing algebra in $d \geq 1$ dimensions

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**Abstract** Non-local representations of the ageing algebra for generic dynamical exponents  $z$  and for any space dimension  $d \geq 1$  are constructed. The mechanism for the closure of the Lie algebra is explained. The Lie algebra generators contain higher-order differential operators or the Riesz fractional derivative. Co-variant two-time response functions are derived. An application to phase-separation in the conserved spherical model is described.

## 1 Introduction: Ageing systems and ageing algebra

Ageing behaviour has been first studied in structural glasses quenched from a molten state to below "glass-transition temperature" by Struik [32]. Nowadays, ageing has been seen in non-equilibrium relaxations in other glassy and non-glassy system far from equilibrium (see e.g. [6, 16] for surveys). Schematically, one may characterise ageing systems by (i) a slow relaxation dynamics, (ii) absence of time-translation-invariance and (iii) dynamical scaling.

In this work<sup>1</sup>, we consider the dynamical symmetries of ageing systems undergoing 'simple ageing', with a dynamics characterised by a single length scale,  $L(t) \sim t^{1/z}$  at large times, which defines the *dynamical exponent*  $z$ . One may ask if the naturally present dynamical scaling in the long-time limit  $t \rightarrow \infty$  can be extended to a larger set of local scale transformation, called '*local scale-invariance*' (LSI). The current state of LSI-theory, with its explicit predictions for two-time responses

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<sup>1</sup> This paper contains the main results from [31], presented by the first author at LT-10.

and correlators, has been recently reviewed in detail in [16]. Here, we describe an algebraic technique in order to extend known representations of LSI algebras with dynamical exponents  $z = 2$  (or  $z = 1$ ) to more general values.

The analysis of the ageing of several simple magnetic systems, without disorder nor frustrations, without any macroscopic conservation law of the dynamics, and undergoing ageing when quenched to a temperature  $T < T_c$  below the critical temperature  $T_c > 0$  is characterised by the dynamical exponent  $z = 2$  [5]. Then, the detailed scaling form of the two-time correlators and responses can be obtained by an extension of simple dynamical scaling with  $z = 2$  towards a larger Lie group [13].

Its Lie algebra is known as ‘ageing algebra’  $\mathfrak{age}(d) = \left\langle X_{0,1}, Y_{\pm\frac{1}{2}}^{(i)}, M_0, R_{ij} \right\rangle_{1 \leq i < j \leq d}$  and can be defined by the following non-vanishing commutators [14]

$$\begin{aligned} [X_n, Y_m^{(i)}] &= \left(\frac{n}{2} - m\right) Y_{n+m}^{(i)}, \quad [X_n, X_{n'}] = (n - n') X_{n+n'}, \quad [Y_{\frac{1}{2}}^{(i)}, Y_{-\frac{1}{2}}^{(j)}] = \delta_{ij} M_0, \\ [R_{ij}, R_{kl}] &= \delta_{il} R_{jk} + \delta_{jk} R_{il} - \delta_{ik} R_{jl} - \delta_{jl} R_{ik}, \quad [R_{ij}, Y_m^{(k)}] = \delta_{jk} Y_m^{(i)} - \delta_{ik} Y_m^{(j)} \end{aligned} \quad (1.1)$$

with  $n, n' = 0, 1, m = \pm\frac{1}{2}$  and  $1 \leq i \leq j \leq d$ . When acting on time-space coordinates  $(t, \mathbf{r})$ , a representation of (1.1) in terms of affine differential operators is:

$$\begin{aligned} X_0 &= -t\partial_t - \frac{1}{2}(\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{x}{2}, \quad X_1 = -t^2\partial_t - t(\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{\mathcal{M}}{2}\mathbf{r}^2 - (x + \xi)t \\ Y_{-\frac{1}{2}}^{(i)} &= -\partial_{r_i}, \quad Y_{\frac{1}{2}}^{(i)} = -t\partial_{r_i} - \mathcal{M}r_i, \quad M_0 = -\mathcal{M} \\ R_{ij} &= r_i\partial_{r_j} - r_j\partial_{r_i} = -R_{ji}. \end{aligned} \quad (1.2)$$

The above representation has a dynamical exponent  $z = 2$  and acts *locally* on the time-space coordinates. Furthermore, it generates a set of dynamical symmetries of the Schrödinger (or diffusion) equation:

$$\hat{S}\phi(t, \mathbf{r}) = \left(2\mathcal{M}\partial_t + \frac{2\mathcal{M}}{t}(x + \xi - d/2) - \nabla_{\mathbf{r}}^2\right)\phi(t, \mathbf{r}) = 0, \quad (1.3)$$

in the sense that each of the generators of  $\mathfrak{age}(d)$  maps a solution of (1.3) onto another solution. The triplet  $(\mathcal{M}, x, \xi)$  characterises the solution  $\phi = \phi_{(\mathcal{M}, x, \xi)}$  of this equation.<sup>2</sup> Furthermore,  $x$  and  $\xi$  are two *independent* scaling dimensions.

For systems undergoing simple ageing with  $z = 2$ , LSI as described by the representation (1.2) of  $\mathfrak{age}(d)$  indeed gives an appropriate description, including several exactly solved examples where  $\xi \neq 0$  is required [14, 16]. The best-known example is the 1D Glauber-Ising model quenched to  $T = 0$ . A main prediction is the form of the two-time (linear) response  $R = R(t, s) = \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0}$  of the order parameter  $\phi$  with respect to its conjugate magnetic field.

In statistical physics, a common formulation uses a stochastic Langevin equation

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<sup>2</sup>  $\mathcal{M} \in \mathbb{R}$  is interpreted as an inverse diffusion constant, or as a non-relativistic mass if  $\mathcal{M} \in i\mathbb{R}$ .

$$\partial_t \phi(t, \mathbf{r}) = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi(t, \mathbf{r})} + \eta(t, \mathbf{r}) \quad (1.4)$$

with a Ginzburg-Landau functional  $\mathcal{H}$  and a centred gaussian noise  $\eta$  with a  $\delta$ -correlated second moment. The standard Janssen-de Dominicis formalism [19, 34] relates this to the equation of motion derived from a dynamical functional  $\mathcal{J}[\tilde{\phi}, \phi] = \mathcal{J}_0[\tilde{\phi}, \phi] + \mathcal{J}_\eta[\tilde{\phi}]$ , written in terms order parameter  $\phi = \phi_{\mathcal{M}, x, \xi}$  and its conjugate *response operator*  $\tilde{\phi} = \tilde{\phi}_{-\mathcal{M}, \tilde{x}, \tilde{\xi}}$  such that the ‘deterministic part’  $\mathcal{J}_0$  is invariant under the action of the Galilei sub-algebra  $\text{gal}(d) = \left\langle Y_{\pm \frac{1}{2}}^{(i)}, M_0, R_{ij} \right\rangle_{1 \leq i < j \leq d}$ . This implies the Bargman super-selection rules [1].

**Theorem.** [28, 16] *All  $n$ -point functions of ‘noisy theory’ described by  $\mathcal{J}$  can be reduced to averages  $\langle \cdot \rangle_0$  calculable from the deterministic part  $\mathcal{J}_0$  alone.*

In particular the response function  $R(t, s) = \left\langle \phi(t) \tilde{\phi}(s) \right\rangle = \left\langle \phi(t) \tilde{\phi}(s) \right\rangle_0$  (see e.g. [19, 34] for introductions and detailed references), is *independent of the noise*  $\eta$  and can be derived from co-variance under  $\text{age}(d)$ . These calculations have been carried out for a long list of models undergoing simple ageing with  $z = 2$  [2, 29, 8, 16].

Can one extend this procedure, at least for linear stochastic Langevin equations of motion, to arbitrary values of the dynamical exponent  $z$ ? If we were to restrict to locally realised algebras, the recent classification of the non-relativistic limits of the conformal algebra [9, 7] would only admit the cases (i)  $z = 1$ : the conformal algebra  $\text{conf}(d)$  or the conformal Galilean algebra  $\text{cga}(d)$  [11, 12, 27], eventually with the exotic central extension for  $d = 2$  [22] (ii)  $z = 2$ : the Schrödinger algebra and (iii)  $z = \infty$ ; all along with their sub-algebras. Further examples can only be found when looking at non-local representation, of known abstract algebras, that is generators more general than first-order linear (affine) differential operators. Some partial information is already available to serve as a guide:

1. the Galilei-invariance of the non-relativistic equation of motion  $\hat{S}\phi = 0$  should be kept (this guarantees the validity of the Bargman superselection rule, hence the applicability of the theorem above):

$$[Y_{\frac{1}{2}}^{(i)}, Y_{-\frac{1}{2}}^{(j)}] = \delta_{ij} M_0, \quad [\hat{S}, Y_{\pm \frac{1}{2}}^{(j)}] = \lambda_{\pm}^{(j)} \hat{S}, \quad (1.5)$$

2. In the context of LSI, different realisations of generalised symmetry algebras have been constructed by using certain fractional derivatives [13, 15, 16]. The closure of these sets of generators can only be achieved by taking a quotient with respect to a certain set of ‘physical’ states. Although this has been successfully applied to certain physical models [3, 8] the closing procedure is not completely determined and it is not clear how to obtain the group (finite) transformations.

A distinct and potentially more promising method has been explored in [17]. Therein, new non-local representations of  $\text{age}(1)$  for an integer-valued dynamical

exponent  $z = n \in \mathbb{N}$  were constructed. This reads

$$\begin{aligned} X_0 &= -\frac{n}{2}t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}, \quad Y_{-\frac{1}{2}} = -\partial_r, \quad M_0 = -\mu \\ Y_{\frac{1}{2}} &= -t\partial_r^{n-1} - \mu r, \quad 2 \leq z = n \in \mathbb{N} \\ X_1 &= \left(-\frac{n}{2}t^2\partial_t - tr\partial_r - (x + \xi)t\right)\partial_r^{n-2} - \frac{1}{2}\mu r^2 \end{aligned} \quad (1.6)$$

The commutation relations (1.1) are satisfied except the following

$$[X_1, Y_{\frac{1}{2}}] = \frac{n-2}{2}t^2\partial_r^{n-3}\hat{S}, \quad (1.7)$$

Consequently, the algebra is 'on shell' algebra that is closed only on quotients with respect to the solution space of the equation

$$\hat{S}\phi(t, r) = \left(z\mu\partial_t - \partial_r^z + \frac{2\mu}{t}\left(x + \xi - \frac{z-1}{2}\right)\right)\phi(t, r) = 0. \quad (1.8)$$

The generators (1.6) act as dynamical symmetries [17] of the equation (1.8), for  $z \in \mathbb{N}$ . In the limit  $z \rightarrow 2$ , the usual representation of the ageing algebra is recovered.

In section 2 we shall generalise the above construction to any spatial dimension  $d \geq 1$ . This transition is not trivial because of non-locality of the generators (1.6). Co-variant two-point functions are computed from these non-local representations in section 3. In section 4, we shall apply these results to some simple physical models, namely the kinetic spherical model with a conserved order-parameter and quenched to  $T = T_c$  and the Mullins-Herring (or Wolf-Villain) equations of interface growth with mass conservation. The time-space responses are calculated from the non-local representations of  $\text{age}(d)$ , to be compared with the known exact results [21, 23, 30, 3]. We conclude in section 5.

## 2 Non-local representations of $\text{age}(d)$ in dimensions $d \geq 1$

It turns out that *only for  $z = 2n$  even, it is possible to extend the non-local representation of ageing algebra (1.6) to  $d \geq 1$  dimensions, while this do not work for  $z = 2n + 1$  odd.* A common treatment of both cases requires the use of the *Riesz fractional derivative* [24, 16]. It is defined as a linear operator  $\nabla_{\mathbf{r}}^\alpha$  acting as follows

$$\nabla_{\mathbf{r}}^\alpha f(\mathbf{r}) = i^\alpha \int_R^d \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\alpha e^{i\mathbf{r} \cdot \mathbf{k}} \hat{f}(\mathbf{k}), \quad (2.1)$$

where the right-hand side as to be understood in a distribution sense and  $\hat{f}(\mathbf{k})$  denotes the Fourier transform. Some elementary properties are: [16]

$$\begin{aligned} \nabla_{\mathbf{r}}^\alpha \nabla_{\mathbf{r}}^\beta &= \nabla_{\mathbf{r}}^{\alpha+\beta}, \quad \nabla_{\mathbf{r}}^2 = \sum_{i=1}^d \partial_i^2 = \Delta_{\mathbf{r}}, \quad [\nabla_{\mathbf{r}}^\alpha, r_i] = \alpha \partial_i \nabla_{\mathbf{r}}^{\alpha-2} \\ [\nabla_{\mathbf{r}}^\alpha, \mathbf{r}^2] &= 2\alpha(\mathbf{r} \cdot \partial_{\mathbf{r}}) \nabla_{\mathbf{r}}^{\alpha-2} + \alpha(d + \alpha - 2) \nabla_{\mathbf{r}}^{\alpha-2}, \quad \nabla_{\mu \mathbf{r}}^\alpha f(\mu \mathbf{r}) = |\mu|^{-\alpha} \nabla_{\mathbf{r}}^\alpha f(\mu \mathbf{r}). \end{aligned}$$

The Riesz fractional derivative can be viewed as a ‘square root’ of the Laplacian.

Now consider the generators:

$$\begin{aligned} X_0 &:= -\frac{z}{2} t \partial_t - \frac{1}{2} (\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{x}{2}, \\ X_1 &:= \left( -\frac{z}{2} t^2 \partial_t - t(\mathbf{r} \cdot \partial_{\mathbf{r}}) - (x + \xi)t \right) \nabla_{\mathbf{r}}^{z-2} - \frac{\mu}{2} \mathbf{r}^2 \\ Y_{-1/2}^{(i)} &:= -\partial_i, \quad Y_{+1/2}^{(i)} := -t \partial_i \nabla_{\mathbf{r}}^{z-2} - \mu r_i, \quad M_0 := -\mu \\ R_{ij} &:= r_i \partial_j - r_j \partial_i = -R_{ji}. \end{aligned} \tag{2.2}$$

The commutators (1.1) of  $\mathfrak{age}(d)$  are seen to hold true, except for

$$[X_1, Y_{\frac{1}{2}}^{(i)}] = \frac{1}{2} (z-2) t^2 \partial_i \nabla_{\mathbf{r}}^{z-4} \hat{S}. \tag{2.3}$$

Hence, the above generators close into a Lie algebra  $\mathfrak{age}(d)$  only in the quotient space over solutions of ‘Schrödinger equation’

$$\hat{S} \phi(t, \mathbf{r}) = \left( z\mu \partial_t - \nabla_{\mathbf{r}}^z + 2\mu t^{-1} \left( x + \xi - \frac{1}{2} (d + z - 2) \right) \right) \phi(t, \mathbf{r}) = 0. \tag{2.4}$$

This representation of  $\mathfrak{age}(d)$  generates dynamical symmetries of the equation (2.4) since  $[\hat{S}, Y_{-\frac{1}{2}}^{(i)}] = [\hat{S}, Y_{\frac{1}{2}}^{(i)}] = [\hat{S}, M_0] = [\hat{S}, R_{ij}] = 0$  and

$$[\hat{S}, X_0] = -\frac{1}{2} z \hat{S}, \quad [\hat{S}, X_1] = -z t \nabla_{\mathbf{r}}^{z-2} \hat{S}.$$

Some comments are in order:

1. the non-locality only enters into the Galilei  $Y_{+\frac{1}{2}}^{(i)}$  and special transformations  $X_1$ . For  $z = 2n$  even, these non-local generators, as well as invariant equation (2.4) are expressed in powers of the Laplacian

$$\begin{aligned} Y_{+1/2}^{(i)} &:= -t \partial_i \Delta_{\mathbf{r}}^{n-1} - \mu r_i \\ X_1 &:= (-n t^2 \partial_t - t(\mathbf{r} \cdot \partial_{\mathbf{r}}) - (x + \xi)t) \Delta_{\mathbf{r}}^{n-1} - \frac{\mu}{2} \mathbf{r}^2, \\ \hat{S} \phi(t, \mathbf{r}) &= \left( 2n\mu \partial_t - \Delta^n + 2\mu t^{-1} \left( x + \xi - \frac{1}{2} (d + 2n - 2) \right) \right) \phi(t, \mathbf{r}) = 0. \end{aligned} \tag{2.5}$$

2. for a dynamical exponent  $z \neq 2n$ , use of the Riesz fractional derivatives (2.2) is necessary and there is no simple relation to the representations of  $\mathfrak{age}(1)$ .

Summarising, the representation of  $\text{age}(d)$  proposed here explicitly uses generators acting non-locally on space. In Fourier space, the generators become local, but non-analytic. The special case of an even-valued dynamical exponent appears to have rather special and possibly non-generic properties.

### 3 Co-variant two-point function

Co-variance under (2.2) gives the two-point function (with  $\phi_i = \phi_{i,(\mu_1, x_1, \xi_1)}(t_i, \mathbf{r}_i)$ )

$$F(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = \langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \rangle \quad (3.1)$$

The result is (with  $\tau = t_1 - t_2, y = t_1/t_2$ ):

$$F = \delta(\mu_1 + \mu_2) t_2^{-\frac{x_1 + x_2}{z}} (y - 1)^{-\frac{2}{z}[\frac{x_1 + x_2}{2} + \xi_1 + \xi_2 - z + 2]} y^{-\frac{1}{z}[x_2 - x_1 + 2\xi_2 - z + 2]} f(|\mathbf{r}|^z \tau^{-1}). \quad (3.2)$$

where  $f$  still has to be found from Galilei-covariance.

**Even dynamical exponent**  $z = 2n$ : If  $p := |\mathbf{r}|^z/\tau$ , Galilei-covariance gives

$$(\tau \partial_{r_j} \Delta_{\mathbf{r}}^{n-1} + \mu r_i) f(p) = r_j \left( (2n)^n p^{\frac{n-1}{n}} \partial_p \Delta_p^{n-1} + \mu \right) f(p) = 0. \quad (3.3)$$

and  $j = 1, \dots, d$ . In particular if  $n = 2$ , a Frobénius series representation leads to

$$\begin{aligned} f(p) = & f_0 {}_0F_2 \left( \frac{1}{2}, \frac{1}{2} + \frac{d}{4}; -\frac{\mu p}{64} \right) + f_1 p^{1/2} {}_0F_2 \left( \frac{3}{2}, \frac{d}{4} + 1; -\frac{\mu p}{64} \right) \\ & + f_2 p^{1/2-d/4} {}_0F_2 (1 - d/4, 3/2 - d/4; -\mu p/64). \end{aligned} \quad (3.4)$$

**Generic dynamical exponent**: matters become simple in Fourier space

$$(\mu \partial_{k_j} + i^{z-2} \tau k_j |\mathbf{k}|^{z-2}) \hat{f}(\tau, \mathbf{k}) = 0 \Rightarrow \hat{f}(\tau, \mathbf{k}) = f_0(\tau) \exp \left[ -\frac{i^{z-2}}{z} \frac{\tau}{\mu} |\mathbf{k}|^z \right] \quad (3.5)$$

This is rewritten in the direct space as follows

$$\begin{aligned} f(\tau, \mathbf{r}) &= \frac{f_0(\tau)}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{k} \exp \left[ i\mathbf{k} \cdot \mathbf{r} - \frac{i^{z-2}}{z} \frac{\tau}{\mu} |\mathbf{k}|^z \right] = \frac{f_0(\tau)}{(2\pi)^d} I_\beta(\mathbf{r}) \\ \beta &:= \alpha \tau = \frac{i^{z-2}}{z\mu} \tau \in \mathbb{C}, \quad I_\beta(\mathbf{r}) := \int_{\mathbb{R}^d} d\mathbf{k} \exp [i\mathbf{k} \cdot \mathbf{r} - \beta |\mathbf{k}|^z] \end{aligned} \quad (3.6)$$

Finally we have (with an infinite radius of convergence for  $z > 1$ )

$$f(\tau, \mathbf{r}) = f_{00} \frac{\Gamma(d/2)}{\Gamma(d/z)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+d}{z})}{n! \Gamma(n + \frac{d}{2})} \left( \frac{\mathbf{r}^2}{4(\alpha \tau)^{2/z}} \right)^n. \quad (3.7)$$

#### 4 Conserved spherical model. Field-theoretical description

The spherical model [4] is defined through spin variable  $S(t, \mathbf{x}) \in \mathbb{R}$ , attached to each site  $\mathbf{x}$  of the hyper-cubic lattice  $\Lambda \subset \mathbb{Z}^d$  and which satisfy the mean spherical constraint  $\langle \sum_{\mathbf{x} \in \Lambda} S(t, \mathbf{x})^2 \rangle = \mathcal{N}$ , where  $\mathcal{N}$  is the number of sites. The Hamiltonian is  $\mathcal{H} = -\sum_{(\mathbf{x}, \mathbf{y})} S_{\mathbf{x}} S_{\mathbf{y}}$ , where the sum is over pairs of nearest neighbours. At equilibrium, a second-order phase transition is observed for  $d > 2$  at some  $T_c > 0$ . The critical exponents have non-mean-field values for  $d < 4$  [20]. The dynamics is given by a Langevin equation with a conserved order parameter (model B) [18]

$$\begin{aligned} \partial_t S(t, \mathbf{x}) &= -\nabla_{\mathbf{x}}^2 [\nabla_{\mathbf{x}}^2 S(t, \mathbf{x}) + \mathfrak{z}(t) S(t, \mathbf{x}) + h(t, \mathbf{x})] + \eta(t, \mathbf{x}) \\ \langle \eta(t, \mathbf{x}) \eta(t', \mathbf{x}') \rangle &= -2T_c \nabla_{\mathbf{x}}^2 \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (4.1)$$

This is a simple but physically reasonable model (since  $\mathfrak{z}(t) \sim 1/t$  for  $t \rightarrow \infty$ ) for the kinetics of phase-separation (for example in alloys). A simple variant is the Mullins-Herring/Wolf-Villain model, where one fixes the Lagrange multiplier  $\mathfrak{z}(t) = 0$ , and which describes the growth of interfaces on a substrate with a conservation of particles along the interface [26, 35]. The correlators and response are studied in detail [21, 23, 10, 30, 3]. Recall the full time-space response in the conserved spherical model for  $d > 4$ , or equivalently in the Mullins-Herring model for any  $d$

$$\begin{aligned} R(t, s; \mathbf{r}) &= \frac{\sqrt{\pi}}{2^{3d/2} \pi^{d/2} \Gamma(d/4)} (t-s)^{-(d+2)/4} \left[ {}_0F_2 \left( \frac{1}{2}, \frac{d}{4}; \frac{\mathbf{r}^4}{256(t-s)} \right) \right. \\ &\quad \left. - \frac{8}{d} \frac{\Gamma(\frac{d}{4} + 1)}{\Gamma(\frac{d}{4} + \frac{1}{2})} \left( \frac{\mathbf{r}^2}{16\sqrt{t-s}} \right) {}_0F_2 \left( \frac{3}{2}, \frac{d}{4} + \frac{1}{2}; \frac{\mathbf{r}^4}{256(t-s)} \right) \right], \end{aligned} \quad (4.2)$$

which we want to compare with the  $\text{age}(d)$ -covariant two-point function, obtained above from the non-local representation (3.4) with  $z = 4$ .

In order to do this, adapt, to the present non-local case, the standard methods of Janssen-de Dominicis theory in non-equilibrium field theory [3], to find a relation between a dynamical symmetry of a deterministic equation with the properties of a solution of a stochastic Langevin equation. The Langevin equation

$$\begin{aligned} \partial_t \phi &= -\frac{1}{4\mu} \nabla_{\mathbf{r}}^2 (-\nabla_{\mathbf{r}}^2 \phi + v(t)\phi + h(t, \mathbf{r})) + \eta \\ \langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle &= -\frac{T_c}{2\mu} \nabla_{\mathbf{r}}^2 \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (4.3)$$

can be viewed as eq. of motion of the Janssen-de Dominicis action, decomposed into deterministic and stochastic parts  $\mathcal{J}(\phi, \tilde{\phi}) = \mathcal{J}_0(\phi, \tilde{\phi}) + \mathcal{J}_{\eta}(\tilde{\phi})$

$$\mathcal{J}_0(\phi, \tilde{\phi}) = \int du d\mathbf{R} \left[ \tilde{\phi} \left( \partial_u - \frac{1}{4\mu} \nabla_{\mathbf{R}}^2 (\nabla_{\mathbf{R}}^2 - v(u)) \right) \phi + h \nabla_{\mathbf{R}}^2 \tilde{\phi} \right] \quad (4.4)$$

$$\mathcal{J}_{\eta}(\tilde{\phi}) = \frac{T}{4\mu} \int du d\mathbf{R} \tilde{\phi}(u, \mathbf{R}) (\nabla^2 \tilde{\phi}(u, \mathbf{R})) + \mathcal{J}_{\text{init}}. \quad (4.5)$$

The averages of an observable  $\mathcal{A}$  is given by the functional integral:

$$\langle \mathcal{A} \rangle = \int \mathcal{D}[\phi] \mathcal{D}[\tilde{\phi}] \mathcal{A}[\phi] \exp(-\mathcal{J}(\phi, \tilde{\phi})) =: \langle \mathcal{A} \exp(-\mathcal{J}_\eta) \rangle_0. \quad (4.6)$$

In particular for the linear response function we obtain<sup>3</sup>

$$\begin{aligned} R(t, s; \mathbf{x} - \mathbf{y}) &:= \left. \frac{\langle \phi(t, \mathbf{x}) \rangle}{\delta h(s, \mathbf{y})} \right|_{h=0} = \langle \phi(t, \mathbf{x}) \nabla_{\mathbf{y}}^2 \tilde{\phi}(s, \mathbf{y}) \exp(-\mathcal{J}_\eta) \rangle_0 \\ &= \nabla_{\mathbf{y}}^2 \langle \phi(t, \mathbf{x}) \tilde{\phi}(s, \mathbf{y}) \exp(-\mathcal{J}_\eta) \rangle_0 = \nabla_{\mathbf{r}}^2 F^{(2)}(t, s; \mathbf{x} - \mathbf{y}), \end{aligned}$$

where  $F^{(2)}(t, s; \mathbf{r})$  is the two-point function, found in section 3 with identification  $\phi = \phi_{\mu, \mathbf{x}, \xi}$  as order parameter and  $\tilde{\phi} = \phi_{-\mu, \tilde{\mathbf{x}}, \tilde{\xi}}$  as response field. In the last line we have used the Bargman super-selection rule [1], which holds in terms of the “mass” parameter  $\mu$ , that is  $\langle \phi_1(t_1, \mathbf{r}_1) \dots \phi_n(t_n, \mathbf{r}_n) \rangle_0 = 0$  unless  $\mu_1 + \dots \mu_n = 0$ . It is enough to consider the case  $\nu = 0$  which gives rise to conserved spherical model for  $d > 4$  and Mullins-Herring model for any  $d$ .

We see that the deterministic part of eq. (4.3) coincides with “Schrödinger equation” for  $z = 4$ , if in addition the time-translation invariance is taken into account (i.e.  $\mathfrak{z}(t) = 0$ ), that is the parameters of non-local representation of the ageing algebra must satisfy  $x + \xi = \tilde{x} + \tilde{\xi} = (d + 2)/2 = 0$ . Then

$$\begin{aligned} R(t, s; \mathbf{r}) &= (t - s)^{-d/4} \nabla_{\mathbf{r}}^2 f \left( \frac{\mathbf{r}^4}{t - s} \right) = (t - s)^{-(d+2)/4} \Delta_p f(p) \\ &= 4(t - s)^{-(d+2)/4} ((d+2)p^{\frac{1}{2}} \partial_p + 4p^{\frac{3}{2}} \partial_p^2) f(p) \\ &= (t - s)^{-(d+2)/4} \times \\ &\quad \times \left[ f'_1 {}_0F_2 \left( \frac{1}{2}, \frac{d}{4}; -\frac{\mu p}{64} \right) + f'_0 p^{1/2} {}_0F_2 \left( \frac{3}{2}, \frac{d}{4} + \frac{1}{2}; -\frac{\mu p}{64} \right) \right. \\ &\quad \left. + f'_2 p^{1-d/4} {}_0F_2 \left( \frac{3}{2} - \frac{d}{4}, 2 - \frac{d}{4}; -\frac{\mu p}{64} \right) \right]. \quad (4.7) \end{aligned}$$

Since the response function must be regular at  $\mathbf{r} = 0$  and vanish for  $|\mathbf{r}| \rightarrow \infty$ , the third term is eliminated, viz.  $f'_2 = 0$ . The constants  $f'_0$  and  $f'_1$  can be related by the known long-term behaviour of the hyper-geometric function [36, 17]. Hence one reproduces the exact result (4.2), but now from the covariance under non-local representation of ageing algebra with dynamical exponent  $z = 4$ .

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<sup>3</sup> In order to compute response function, we must introduce small perturbation  $h$  (conjugate magnetic field) in the right-hand side of the eq. (4.3), which respects the conservation law. This generates respectively an additional term in the Janssen-de Dominicis action, which we have written explicitly (4.4).



## 5 Conclusions

When trying to construct a closed Lie algebra for generalised scale-transformations with an arbitrary dynamical exponent  $z \in \mathbb{R}$ , we have been led to consider non-local representations of the ageing algebra  $\text{age}(d)$ , for general  $d \geq 1$  [17, 31].

It was necessary to slightly extend the usual definition of the notion of *dynamical symmetry*. Conventionally, the infinitesimal generator  $X$  of a dynamical symmetry of the equation of motion  $\hat{S}\phi = 0$  must satisfy  $[\hat{S}, X] = \lambda_X \hat{S}$  as an operator, where  $\lambda_X$  should be a scalar or a function. Here,  $\lambda_X$  may be an *operator* itself. The Lie algebra closes on the quotient space with respect to  $\hat{S}\phi = 0$ .

Several details depend on the value of  $z$ .

1. For an odd dynamical exponent  $z \geq 2$ , the generalisation from the one-dimensional case requires the explicit introduction of some kind of fractional derivative. For our purposes, the Riesz fractional derivative turned out to have the required algebraic properties. In addition, the result derived for the co-variant two-point function is compatible with the directly treatable case when  $z$  is even, but we are not aware of confirmed physical applications in this case.
2. For  $z$  even, the algebra (2.5) contains  $d + 1$  non-local generators of generalised Galilei-transformation and special transformations, constructed with linear differential operators of order  $z - 1$ . By analogy with the  $1D$  case [17], we suspect that these might be interpreted as generating transformation of distribution functions of the positions, rather than *bona fide* coordinate transformations. The example studied here (conserved spherical model for  $d > 4$  or equivalently in the Mullins-Herring equation for any  $d$ ) might be the first step towards an understanding how to use such non-local transformations in applications to the non-equilibrium physics of strongly interacting particles.

Extensions to more general representations may be of interest [25].

Recall that in the context of interface growth with conserved dynamics, exactly the kind of non-local generalised Galilei-transformation we have studied here has already been introduced in analysing the stochastic equation (related to molecular beam epitaxy (MBE)), with constants  $\nu$ ,  $\lambda$  and a white noise  $\eta$

$$\partial_t \phi = -\nabla^2 \left[ \nu \nabla^2 \phi + \frac{\lambda}{2} (\nabla \phi)^2 \right] + \eta \quad (5.1)$$

It can be shown that Galilei-invariance leads to a non-trivial hyper-scaling relation, expected to be exact [33]. In particular, they obtain  $z = 4$  in  $d = 2$  space dimensions. We hope to return to a symmetry analysis of these non-linear equations in the future. In any case, the available evidence that generalised Galilei-invariance could survive the loop expansion is very encouraging.

**Acknowledgements:** Ce travail a reçu du support financier par PHC Rila et par le Collège Doctoral franco-allemand Nancy-Leipzig-Coventry (Systèmes complexes à l'équilibre et hors équilibre) de l'UFA-DFH.

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